Year 13 Mathematics

IAS 3.3

Trigonometry

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Contents

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NCEA 3 Internal Achievement Standard 3.3 – Trigonometry

This achievement standard involves applying trigonometric methods in solving problems.

Achievement	Achievement with Merit	Achievement with Excellence
• Apply trigonometric methods in solving problems.	• Apply trigonometric methods, using relational thinking, in solving problems.	• Apply trigonometric methods, using extended abstract thinking, in solving problems.

◆ This achievement standard is derived from Level 8 of The New Zealand Curriculum and is related to the achievement objectives
  - ❖ manipulate trigonometric expressions
  - ❖ form and use use trigonometric equations

  in the Mathematics strand of the Mathematics and Statistics Learning Area.

◆ Apply trigonometric methods in solving problems involves:
  - ❖ selecting and using methods
  - ❖ demonstrating knowledge of concepts and terms
  - ❖ communicating using appropriate representations.

◆ Relational thinking involves one or more of:
  - ❖ selecting and carrying out a logical sequence of steps
  - ❖ connecting different concepts or representations
  - ❖ demonstrating understanding of concepts
  - ❖ forming and using a model;

  and also relating findings to a context, or communicating thinking using appropriate mathematical statements.

◆ Extended abstract thinking involves one or more of:
  - ❖ devising a strategy to investigate or solve a problem
  - ❖ identifying relevant concepts in context
  - ❖ developing a chain of logical reasoning, or proof
  - ❖ forming a generalisation;

  and using correct mathematical statements, or communicating mathematical insight.

◆ Problems are situations that provide opportunities to apply knowledge or understanding of mathematical concepts and methods. Situations will be set in real-life or mathematical contexts.

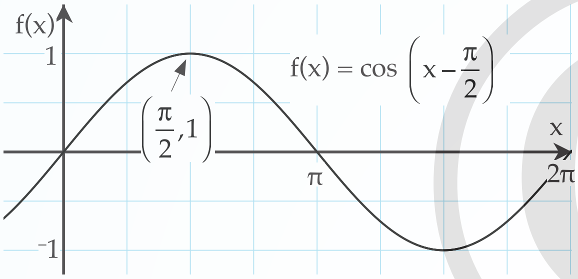
◆ Methods include a selection from those related to:
  - ❖ trigonometric identities
  - ❖ reciprocal trigonometric functions
  - ❖ properties of trigonometric functions
  - ❖ solving trigonometric equations
  - ❖ general solutions.

Example

Graph $f(x) = \cos\left(x - \frac{\pi}{2}\right)$ from 0 to 2π.

We draw $f(x) = \cos x$ from the translated origin at $\left(\frac{\pi}{2}, 0\right)$.

The graph $y = \cos x$ starts at (0, 1) so we draw our graph starting at $\left(\frac{\pi}{2}, 1\right)$.



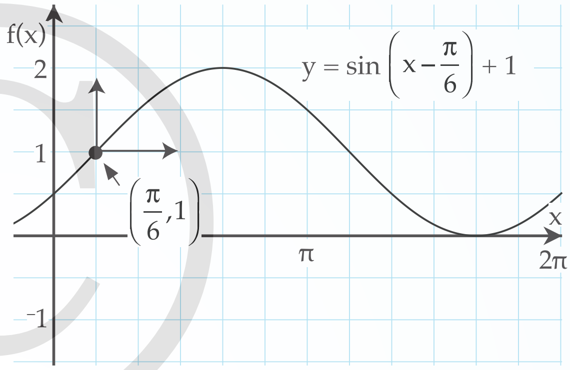
Note: The final graph of $f(x) = \cos\left(x - \frac{\pi}{2}\right)$ is the graph $y = \sin x$.

Example

Graph $y = \sin\left(x - \frac{\pi}{6}\right) + 1$ from 0 to 2π.

This has two translations, across $\frac{\pi}{6}$ and up 1.

We draw $y = \sin x$ from the translated origin at $\left(\frac{\pi}{6}, 1\right)$.



After the sine curve is drawn from $\left(\frac{\pi}{6}, 1\right)$, the graph is extended back.

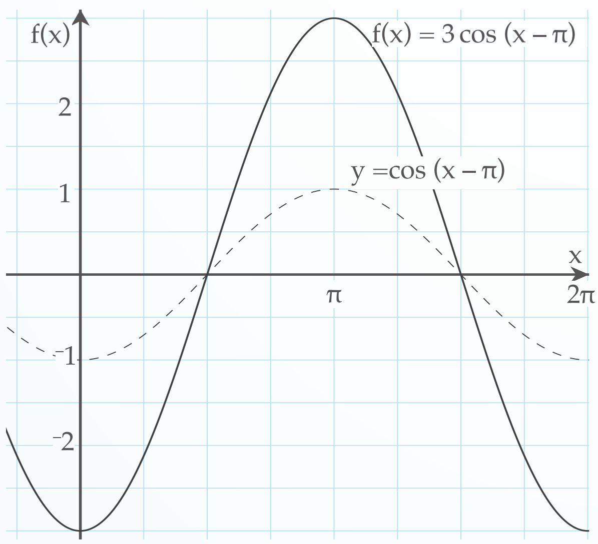
Example

Graph $f(x) = 3\cos(x - \pi)$ from 0 to 2π.

We draw $f(x) = \cos x$ from the translated origin at $(\pi, 0)$ then extend it back.

This translated graph is then stretched three times vertically.

Just go to the key points and move them three times as far from the centre line.



Example

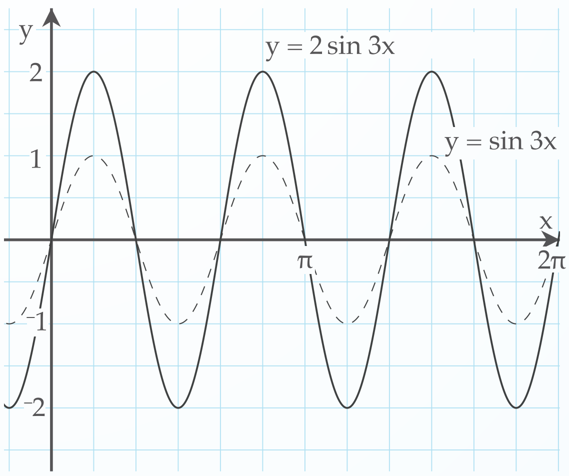
Graph $y = 2\sin 3x$ from 0 to 2π.

We draw $y = \sin x$ but with a $\frac{1}{3}$ horizontal scale.

Where we normally would have a maximum at $\left(\frac{\pi}{2}, 1\right)$ we now have a maximum at $\left(\frac{\pi}{6}, 2\right)$.

We continue this graph until 2π checking to make sure we have three complete cycles.

Now we stretch each point vertically two times.

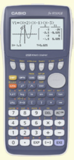


For these Examples, three different approaches are presented for finding a particular solution. Students should concentrate on the approach they find easiest to understand.
Approach 1 (in blue) is manipulation of the equation, approach 2 (in pink) is by graphing the problem and approach 3 (in yellow) is using the solver built into the graphics calculator.

Example Find one solution to the equation $4\sin\left(\frac{x}{2}\right) = {}^{-}1.234$

Manipulation of the equation.

$$4\sin\left(\frac{x}{2}\right) = {}^{-}1.234$$
$$\sin\left(\frac{x}{2}\right) = {}^{-}0.3085$$
$$\frac{x}{2} = \sin^{-1}({}^{-}0.3085)$$
$$x = {}^{-}0.313\,62 \times 2$$
$$= {}^{-}0.6272 \qquad \text{(4 sf)}$$



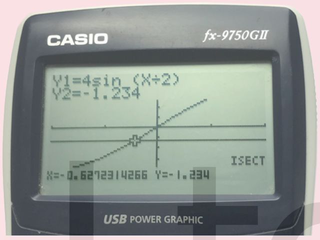
Using the solver on the Casio 9750GII. Check your calculator is set to radians then enter the equation $4\sin\left(\frac{x}{2}\right) = {}^{-}1.234$.

MENU 8 (EQUA) F3 4 sin (X ÷ 2) SHIFT . (=) (–) 1 . 2 3 4 EXE F6 Getting a soln. x = ⁻0.6272.

The graphical approach on the Casio 9750GII. Draw the graph of $y = 4\sin\left(\frac{x}{2}\right)$ and $y = {}^{-}1.234$ from 0 to 2π. We can see from the graph that these do not intersect from 0 to 2π so in the view window we change the domain to ⁻π to π then select and graph solve (G-Solv) and then intersection (ISCT).

SHIFT F3 (V Window) (-) (Xmin) SHIFT EXP (π) EXE SHIFT (max) EXP (π) EXE EXE F6 (DRAW) SHIFT F5 (G-Solv) F5 (ISCT)

The calculator will find the solution x = ⁻0.6272.



Example Find one solution to the equation $20 - 25\cos\left(x + \frac{\pi}{2}\right) = 40$

Manipulation of the equation.

$${}^{-}25\cos\left(x+\frac{\pi}{2}\right) = 20$$
$$\cos\left(x+\frac{\pi}{2}\right) = {}^{-}0.8$$
$$x + \frac{\pi}{2} = \cos^{-1}({}^{-}0.8)$$
$$x = 2.4981 - \frac{\pi}{2}$$
$$x = 0.9273 \qquad \text{(4 sf)}$$

To use the solver on the TI-84 calculator the equation must be in the form 0 = equation.

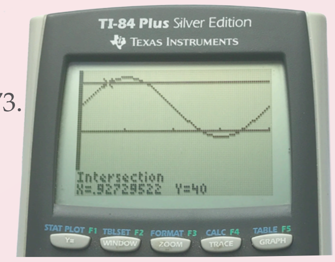
$$0 = 20 + 25\cos\left(x + \frac{\pi}{2}\right)$$

MATH Solver ENTER ▲ CLEAR 2 0 + 2 5 COS X + 2ND ^ (π) ÷ 2) ENTER Enter a guess (e.g. 0) and ALPHA ENTER (SOLVE) to get x = 0.9273.

The graphical approach on the TI-84 Plus. Draw the graphs of $y = 40$ and $y = 20 - 25\cos\left(x+\frac{\pi}{2}\right)$ from x = 0 to x = 2π and adjust the y values to go from ⁻50 to 50 as this graph has a large amplitude.

WINDOW ▼ ▼ ▼ (–) 5 0 ENTER 5 0 ENTER GRAPH 2ND TRACE (CALC) 5 (intersect)

You will be asked to select the curves that are to intersect and to enter a guess near each point of intersection (or use the cursor keys to move the point close to the intersection).
The solution is x = 0.9273.



Modelling Practical Situations using Trigonometric Functions

Modelling Practical Situations using Trigonometric Functions

If you have information about a problem that is best modelled by a trigonometric function, then we can use our understanding of amplitude, frequency and period to write an equation and subsequently solve this equation.

A periodic function that follows a sine curve will have an equation f(x) = A sin B (x + C) + D. The constants A, B, C and D were defined previously. When we are modelling a practical situation we are likely to know the maximum, minimum values along with the period and horizontal shift. We use these data values to find the constants A, B, C and D.

The amplitude A is given by

$$A = \frac{(\text{maximum} - \text{minimum})}{2}$$

The horizontal stretch B is given by

$$\text{period} = \frac{2\pi}{B}$$

$$B = \frac{2\pi}{\text{period}}$$

where the period is the x value (often time) before the graph starts to repeat itself.

The horizontal translation or shift $^{-}$C is the distance along the x axis from x = 0 (or t = 0) until the sine curve starts. Alternatively it is the time to the maximum point minus a quarter of the period.

$$C = {}^{-}\text{shift}$$

$$= {}^{-}(\text{time to maximum} - \tfrac{1}{4}\text{period})$$

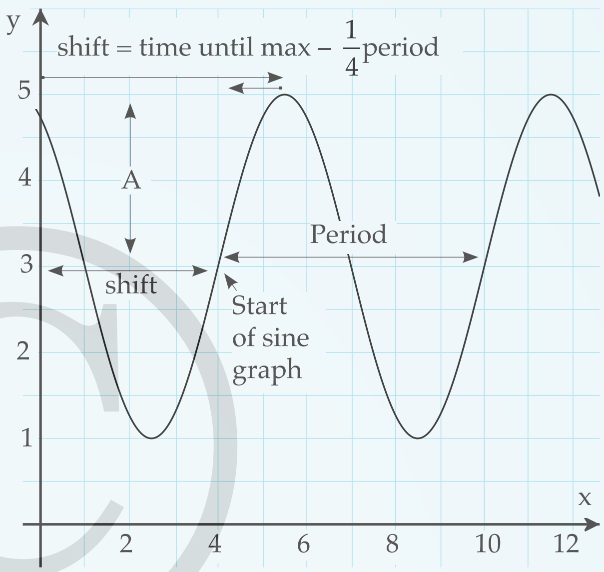
OR

$$= {}^{-}\text{time to maximum} + \tfrac{1}{4}\text{period}$$

The vertical translation D is defined as the average height.

$$D = \frac{(\text{maximum} + \text{minimum})}{2}$$

If we study the graph shown here we can see the standard sine curve starts at the point (4, 3). It has a maximum value of 5 and a minimum value of 1 with a period of 6. The shift from the start is $^{+}$4.



Therefore we can find the equation by finding each constant A, B, C and D.

$$A = \frac{(\text{maximum} - \text{minimum})}{2}$$

$$= \frac{5-1}{2}$$

$$= 2$$

$$B = \frac{2\pi}{\text{period}}$$

$$= \frac{2\pi}{6}\left(\frac{\pi}{3}\right)$$

$$C = {}^{-}\text{shift}$$

$$= {}^{-}(\text{time to maximum} - \tfrac{1}{4}\text{period})$$

OR

$$= {}^{-}\text{time to maximum} + \tfrac{1}{4}\text{period}$$

$$= {}^{-}4$$

$$D = \frac{(\text{maximum} + \text{minimum})}{2}$$

$$= \frac{(5+1)}{2}$$

$$= 3$$

The equation is therefore $y = 2\sin\frac{2\pi}{6}(x-4) + 3$

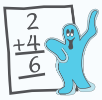
Summary: **If T = period then**

$$A = \frac{\text{max - min}}{2}$$

$$B = \frac{2\pi}{T}$$

$$C = {}^{-}\text{time to max} + \frac{1}{4}\text{ period} \quad \text{OR} \quad {}^{-}\text{time to max} + \frac{T}{4}$$

$$D = \frac{\text{max} + \text{min}}{2}$$

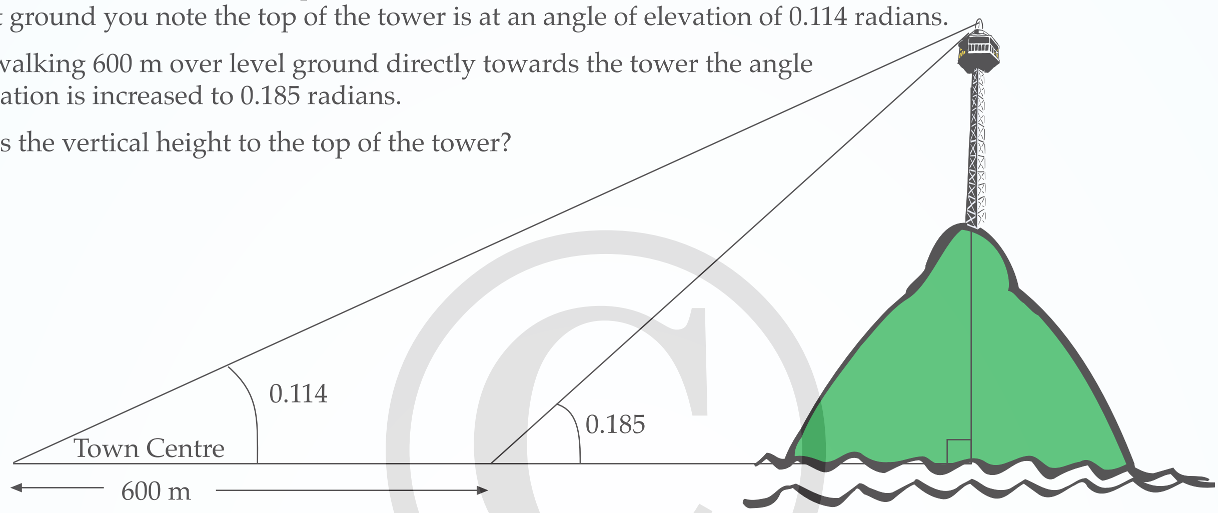


Example

A communication tower is on top of an island in a river.
On flat ground you note the top of the tower is at an angle of elevation of 0.114 radians.

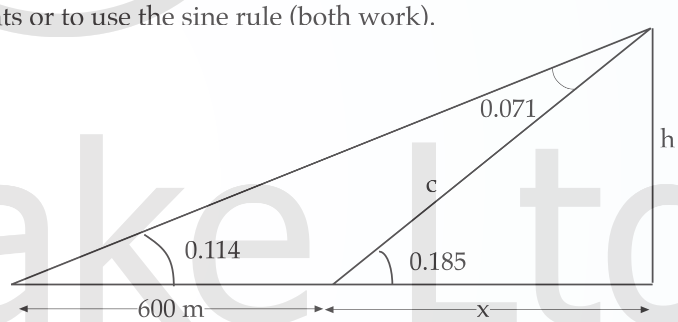
After walking 600 m over level ground directly towards the tower the angle of elevation is increased to 0.185 radians.

What is the vertical height to the top of the tower?



Redraw the diagram letting the height be h and the distance to the base (not required) be x.

The two approaches explained here are to use tangents or to use the sine rule (both work).



Using tangents

$$\tan\theta = \frac{\text{opp}}{\text{adj}}$$

$$\tan 0.114 = \frac{\text{height}}{\text{horizontal distance}}$$

Substitute height = h and horizontal dist = 600 + x

$$= \frac{h}{x + 600}$$

Similarly with the small triangle

$$\tan 0.185 = \frac{\text{height}}{\text{horizontal distance}}$$

Substitute height = h and horizontal dist = x

$$\tan 0.185 = \frac{h}{x}$$

As we do not want x we rearrange one equation so x becomes the subject and substitute it into the other.

$$x \tan 0.185 = h$$

$$x = \frac{h}{\tan 0.185}$$

Substitute in the first equation and solve

$$\tan 0.114 = \frac{h}{\frac{h}{\tan 0.185} + 600}$$

$$h = \left(\frac{h}{\tan 0.185} + 600\right) \tan 0.114$$

$$8.734\,h = 5.344\,h + 600$$

$$h = 177 \text{ m} \quad (3 \text{ sf})$$

Using the sine rule we use the angle properties of a triangle to find the angle at the top and then calculate length c.

$$\text{angle at top} = 0.185 - 0.114$$

$$= 0.071$$

External angle of a triangle = sum 2 opposite internal angles.

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

$$\frac{600}{\sin 0.071} = \frac{c}{\sin 0.114}$$

$$c = \frac{600 \times \sin 0.114}{\sin 0.071}$$

$$c = 962.103$$

Now with the small triangle we can calculate h.

$$\sin 0.185 = \frac{\text{opp}}{\text{hyp}}$$

$$\sin 0.185 = \frac{h}{962.103}$$

$$h = 962.103 \sin 0.185$$

$$h = 177 \text{ m} \quad (3 \text{ sf})$$

143. A plane is sighted at point A from a radar station at O. The plane is at an angle of elevation of 44° and at a height of 6800 metres. Some time later the same plane is sighted at point B from the radar station at an angle of elevation of 27° and at a height of 4900 metres. C and D are points on the ground vertically below points A and B and angle COD is 48°.

a) Draw a diagram to represent the situation described above.

Remember to label all applicable points, angles and lengths.

b) Calculate the distance between the points C and D to three significant figures.

Draw a two-dimensional diagram here.

144. A large tree stands at the end of a road. From one particular point on the road, the top of the tree is at angle A degrees. After walking directly towards the tree along a level road for 23 metres the angle of elevation is increased to B degrees.

a) Show that h, the height of the tree can be represented by $h = \dfrac{23}{\cot A - \cot B}$

b) If angle A is 27° and angle B is 85° find the height of the tree to the nearest metre.

Draw a two-dimensional diagram here.

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97. LHS = sin(x + y).sin(x – y)

$= (\sin x\cos y + \sin y\cos x)(\sin x\cos y - \sin y\cos x)$

$= \sin^2 x.\cos^2 y - \sin^2 y.\cos^2 x$

$= \sin^2 x.(1 - \sin^2 y) - \sin^2 y.(1 - \sin^2 x)$

$= \sin^2 x - \sin^2 x.\sin^2 y - \sin^2 y + \sin^2 y.\sin^2 x$

$= \sin^2 x - \sin^2 y$

= RHS

98. $\text{LHS} = \tan\left(A - \frac{\pi}{4}\right)\tan\left(A + \frac{\pi}{4}\right)$

$= \frac{\left(\tan A - \tan\frac{\pi}{4}\right)\left(\tan A + \tan\frac{\pi}{4}\right)}{\left(1 + \tan A\tan\frac{\pi}{4}\right)\left(1 - \tan A\tan\frac{\pi}{4}\right)}$

$= \frac{(\tan A - 1)(\tan A + 1)}{(1 + \tan A)(1 - \tan A)}$

$= \frac{^-(1 - \tan A)}{(1 - \tan A)}$

$= {}^-1$

= RHS

99. LHS = cot(A + B)

$= \frac{1}{\tan(A + B)}$

$= \frac{1 - \tan A\tan B}{\tan A + \tan B}$

Divide top and bottom by tan A.tan B

$= \frac{\frac{1}{\tan A\tan B} - \frac{\tan A\tan B}{\tan A\tan B}}{\frac{\tan A}{\tan A\tan B} + \frac{\tan B}{\tan A\tan B}}$

$= \frac{\cot A\cot B - 1}{\cot B + \cot A}$

= RHS

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100. $\cos^2 4x - 0.5 = 0.5(2\cos^2 4x - 1)$

$= 0.5\cos(8x)$

101. $\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) = 0.5\sin x$

102. LHS = $(\sin A - \cos A)^2$

$= \sin^2 A - 2\sin A\cos A + \cos^2 A$

= 1 – 2 sin A cos A

= 1 – sin 2A

= RHS

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103. $\text{RHS} = \frac{2\cos 2x}{\sin 2x}$

$= \frac{2(\cos^2 x - \sin^2 x)}{2\sin x\cos x}$

$= \frac{\cos^2 x - \sin^2 x}{\sin x\cos x}$

$= \frac{\cos x}{\sin x} - \frac{\sin x}{\cos x}$

= cot x – tan x

= LHS

104. sin 3A = sin(A + 2A)

= sin A cos 2A + cos A sin 2A

$= \sin A(1 - 2\sin^2 A) + 2\sin A\cos^2 A$

$= \sin A - 2\sin^3 A + 2\sin A(1 - \sin^2 A)$

$= 3\sin A - 4\sin^3 A$

105. cos 3A = cos (A + 2A)

= cos A cos 2A – sin A sin 2A

$= \cos^3 A - \cos A\sin^2 A - 2\sin^2 A\cos A$

$= \cos^3 A - 3\cos A\sin^2 A$

$= 4\cos^3 A - 3\cos A$

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106. $x = 2n\pi \pm 2.0944$
or $x = n\pi + (^-1)^n 0.3398$
x = 0.3398, 2.0944, 2.8018, 4.1888

107. $x = n\pi + (^-1)^n 0.5236$
or $x = n\pi - (^-1)^n 0.7297$
x = 0.5236, 2.6180, 3.8713, 5.5535

108. $x = 2n\pi \pm 0.8411$
or $x = 2n\pi \pm 2.4189$

x = 0.8411, 2.4189, 3.8643, 5.4421

109. $x = n\pi + 1.2490$
or $x = n\pi - 1.1071$

x = 1.2490, 2.0344, 4.3906, 5.1760

110. (2sin x – 1)(sin x + 2) = 0
$x = n\pi + (^-1)^n 0.5236$ only
x = 0.5236, 2.6180

111. $x = 2n\pi \pm 0.8411$
or $x = 2n\pi \pm 3.1416$
x = 0.8411, 3.1416, 5.4421

112. tan x = 0
x = 0, π, 2π

113. $x = n\pi + (^-1)^n 0.3398$
or $x = n\pi - (^-1)^n 0.2526$
x = 0.3398, 2.8018, 3.3943, 6.0305

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114. P = 4 sin 9θ + 4 sin θ

115. $Q = \frac{1}{2}(\cos 4A + \cos(2A + 2B))$

116. R = 5 cos 10θ + 5 cos 2θ

117. S = 2(cos(3A + 2B) – cos 5A)

118. T = 3(cos 2x – cos 4x)

119. $U = \cos\left(2x + \frac{\pi}{2}\right) + \cos\frac{\pi}{2}$

$= \cos\left(2x + \frac{\pi}{2}\right)$

120. $V = \sin 4x + \sin\frac{\pi}{2}$

= sin 4x + 1

121. W = 0.5 cos π – 0.5 cos 2x

$= {}^-0.5 - 0.5\cos 2x$

122. X = 3 (cos 2π + cos 2x)

= 3 + 3 cos 2x

123. $Y = 5(\cos{}^-2x - \cos\left(\frac{\pi}{4} + \frac{\pi}{4}\right))$

= 5 cos 2x

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124. $4\left(\cos\left(\frac{\pi}{3}\right) - \cos(2x)\right) = 4$

$2x = 2n\pi \pm 2.0944$

$x = n\pi \pm 1.0472$

x = 1.0472, 2.0944, 4.1888, 5.2359

or $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

125. cos 2x = 1

2x = 2nπ

x = nπ

x = 0, π, 2π

126. sin(2x – 0.5) + sin (0.5) = 2 x 0.4567

$2x = n\pi + (^-1)^n 0.4489 + 0.5$

$x = 0.5n\pi + (^-1)^n 0.2244 + 0.25$

x = 0.4744, 1.596, 3.616, 4.738

127. $2x = n\pi + (^-1)^n 0.4058 - 0.5$

$x = 0.5n\pi + (^-1)^n 0.2029 - 0.25$

x = 1.118, 3.095, 4.259, 6.236

128. $2(\cos(4x + \pi) + \cos\pi) = {}^-3$

$4x + \pi = 2n\pi \pm 2.0944$

$x = 0.5n\pi \pm 0.5236 - 0.7854$

x = 0.2618, 1.3090, 1.8326, 2.8798, 3.4034, 4.4506, 4.9742, 6.0214

129. $(2x - 0.5267) = 2n\pi \pm 1.8140$

$x = n\pi \pm 0.9070 + 0.2634$

x = 1.1703, 2.4980, 4.3120, 5.6396